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OPTIMUM COPLANAR FLIGHTS BETWEEN ORBITS

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ABSTRACT

The optimum coplanar flight between assigned orbits is considered in this work. It is realized with the aid of a boost with minimum value of characteristic velocity. This ensures the minimum fuel consumption. It is postulated that the motion on all orbits, whether initial, intermediary or final, is Keplerian. Two cases are considered, the flight between circular orbits and that between orbits of small eccentricity. The cases of flight without taking into account the motion time, and those when account of concrete motion time is taken, are dealt with separately.

In the first approximation it is possible to relate these problems to those of space probes to Mars, Venus and other planets of the solar system, as , for example, the flight of the interplanetary automatic station "Mars-1". The problems linked with AES orbit variations, and, in particular, the coplanar problem of hitting another satellite from an assigned AES orbit by means of a unique impulse (boost), minimum from the standpoint of mass consumption, also belong to that category.

* * *

#1. - MINIMUM OF THE FUNCTION DEFINED BY A FAMILY OF VARIABLES WITH TWO DISCONTINUITIES

Let $$y_i = y_i(x), \quad i = 1, 2, \ldots, n \tag{1,1}$$

be the unknown function of an independent variable $x \in [a, b]$, continuous everywhere with the partial derivatives with the exception of two ordinary discontinuities of the first kind at two points x_1, x_2, unknown beforehand.

Whenever they are continuous, the functions (1,1) must everywhere satisfy the system of differential equation

$$\frac{dy_i}{dx}=f_i(x, y_j), \quad i, j=1, 2, \dots, n, \tag{1,2}$$

assuming at the same time that the conditions of theorem existence and uniqueness are fulfilled.

We shall introduce the following denotations:

$$\left.\begin{array}{lll} y_{i1a}=y_i(a), & y_{i1}=y_i(x_1+0), & y_{i22}=y_i(x_2+0), \\ y_{i11}=y_i(x_1-0), & y_{i2}=y_i(x_2-0), & y_{i2b}=y_i(b). \end{array}\right\} \tag{1,3}$$

For the values of variables (1,1) at end points a, b and at points of discontinuity x_1, x_2 there is the system of conditions

$$\tilde{\varphi}_s(a, b, x_1, x_2, y_{i1a}, y_{i11}, y_{i1}, y_{i2}, y_{i22}, y_{i2b})=0, \\ s=1, 2, \dots, r<3n+4. \tag{1,4}$$

Let us consider the assigned function

$$g=\tilde{g}(a, b, x_1, x_2, y_{i1a}, y_{i11}, y_{i1}, y_{i2}, y_{i22}, y_{i2b}). \tag{1,5}$$

Functions (1,1) must be found, satisfying both, the differential equations (1,2) and the conditions (1,4), and also the end points a, b and the discontinuity points x_1, x_2 so that the function g have a minimum.

We shall provide the solution of the variational problem posed, if the general solution of the system (1,2) is known:

$$\Phi_i=y_i-y_i(x, c_j)=0, \quad i, j=1, 2, \dots, n, \tag{1,6}$$

where c_j are arbitrary constants. We shall estimate, that the functions Φ_i have continuous partial derivatives over all the arguments, and that the determinant

$$\left|\frac{\partial\Phi_i}{\partial c_j}\right| \neq 0. \tag{1,7}$$

Denoting

$$\left.\begin{array}{lll} y_{i1}(x, c_{j1}) = y_i(x) & \text{at} & x \in |a, x_1|, \\ y_i(x, c_j) = y_i(x) & \text{at} & x \in |x_1, x_2|, \\ y_{i2}(x, c_{j2}) = y_i(x) & \text{at} & x \in |x_2, b|, \end{array}\right\} \qquad (1,8)$$

we have:

$$\left.\begin{array}{lll} y_{i1a} = y_{i1}(a, c_{j1}), & y_{i1} = y_i(x_1, c_j), & y_{i22} = y_{i2}(x_2, c_{j2}), \\ y_{i11} = y_{i1}(x_1, c_{j1}), & y_{i2} = y_i(x_2, c_j), & y_{i2b} = y_{i2}(b, c_{j2}). \end{array}\right\} \qquad (1,9)$$

As a consequence of correlations (1,9), the conditions (1,4) will be

$$\varphi_s(a, b, x_1, x_2, c_{j1}, c_j, c_{j2}) = 0, \qquad (1,10)$$

and the function (1,5) will take the form

$$g = g(a, b, x_1, x_2, c_{j1}, c_j, c_{j2}). \qquad (1,11)$$

The earlier-formulated problem is reduced in this case to the equivalent problem of finding the minimum of the function of finite number of variables. What is required is the determination of the function (1,8), i. e. the finding of the value of arbitrary constants c_{j1}, c_j, c_{j2}, and of the points a, b, x_1, x_2, satisfying the system of conditions (1,10) in such a way, that the function (1,11) have a minimum.

If at extreme values of arguments the functions (1,10) and (1,11) have continuous partial derivatives of the first order, then, as is well known [1], it is necessary that there exist such constant multipliers $\lambda_1, \lambda_2, \dots, \lambda_r$, for which the conditions

$$\frac{\partial g}{\partial a} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial a} = 0, \quad \frac{\partial g}{\partial b} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial b} = 0, \qquad (1,12)$$

$$\frac{\partial g}{\partial x_1} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial x_1} = 0, \quad \frac{\partial g}{\partial x_2} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial x_2} = 0, \qquad (1,13)$$

$$\frac{\partial g}{\partial c_{j1}} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial c_{j1}} = 0, \quad \frac{\partial g}{\partial c_{j2}} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial c_{j2}} = 0, \qquad (1,14)$$

$$\frac{\partial g}{\partial c_j} + \sum_{s=1}^{r} \lambda_s \frac{\partial \varphi_s}{\partial c_j} = 0, \quad j = 1, 2, \dots, n. \qquad (1,15)$$

are fulfilled. Moreover, if by its arguments, the rank of the matrix constituted of partial derivative functions $\varphi_S(a, b, x_1, x_2, c_{j1}, c_j, c_{j2})$ is equal to r, the multipliers $\lambda_1, \lambda_2, \ldots, \lambda_r$ are unique.

Therefore a system of necessary conditions (1,12) — (1,15), (1,10) is obtained from 3n + r + 4 equations with the very same number of unknowns $a, b, x_1, x_2, c_{j1}, c_j, c_{j2}, \lambda_s$ for the solution of the problem set up.

In the case of a problem with fixed ends, i. e. at values a, b, y_{ha}, y_{hb}, fixed beforehand, the constants c_{j1}, c_{j2} will be found unilaterally, as a consequence of the correlation (1,7), and the equations (1,12) and (1,14) in the system of indispensable conditions (1,10) and (1,12)— (1,15), must be rejected.

#2.- SYSTEM OF NECESSARY CONDITIONS IN CASE OF A FLIGHT WITHOUT TAKING INTO ACCOUNT THE MOTION TIME IN ORBITS

We shall define the complanar Keplerian orbits by the following elements:

$$p = \frac{1}{\sqrt{a(1-e^2)}}, \quad q = \frac{e}{\sqrt{a(1-e^2)}}, \quad \omega. \tag{2,1}$$

where a, e, ω are respectively the major semi-axis, the eccentricity and the angular distance from the pericenter to the polar axis. The polar angle ϑ will be taken as the independent variable.

Assume that the initial orbit has the elements p_1, q_1, ω_1, and the final one — p_2, q_2, ω_2. The impulse, as a result of which we obtain the intermediate orbit of the flight with the elements p, q, , takes place at the time when the polar angle is $_1$. At $\vartheta = \vartheta_2$ this orbit will intersect the final orbit. We now must find the flight orbit for assigned initial and final orbits, i. e. its elements p, q, ω and also the angles ϑ_1 and ϑ_2, in such a way, that the value of the characteristic velocity of the initial impulse be at the minimum.

Since the elements of the boundary orbits p_1, q_1, ω_1, and p_2, q_2, ω_2 are assigned, we shall utilize for the system of necessary conditions the equations (1,13) and (1,15), in which x_1 and x_2 play the role of ϑ_1 and ϑ_2, and $p_1, q_1, \omega_1, p, q, \omega, p_2, q_2, \omega_2$ are respectively taken for

At elements' disconitinuity points the following conditions must be fulfilled:

$$\varphi_1 = p^2 + pq\cos(\vartheta_1 - \omega) - p_1^2 - p_1 q_1 \cos(\vartheta_1 - \omega_1) = 0, \qquad (2,2)$$

$$\varphi_2 = p^2 + pq\cos(\vartheta_2 - \omega) - p_2^2 - p_2 q_2 \cos(\vartheta_2 - \omega_2) = 0, \qquad (2,3)$$

which, as a consequence of correlations

$$\frac{1}{a(1-e^2)} + \frac{e\cos(\vartheta - \omega)}{a(1-e^2)} = p^2 + pq\cos(\vartheta - \omega) \qquad (2,4)$$

denote the continuity of the values of radius-vectors at these points.

We shall use for the function g

$$g = \frac{1}{2K^2}\left[(V_{r_1} - V_r)^2 + (V_{\vartheta_1} - V_\vartheta)^2\right], \qquad (2,5)$$

where V_r, V_ϑ and V_{r_1}, V_{ϑ_1} are respectively the radial and transverse vector components at the point of the initial impulse, prior and after the latter, and

$$K = k\sqrt{M}. \qquad (2,6)$$

In the latter correlation k is the gravitational constant, M is the mass of the central body M. We neglect the mass of the rocket by comparison with that of the central body, and also the masses of planets by comparison with that of the Sun, whenever we analyze interplanetary flights.

For velocity components, we have:

$$V_r = \frac{Ke\sin(\vartheta_1 - \omega)}{\sqrt{a(1-e^2)}} = Kq\sin(\vartheta_1 - \omega), \quad V_\vartheta = \frac{K\sqrt{a(1-e^2)}}{r_1} = \frac{K}{pr_1} \qquad (2,7)$$

and analogously

$$V_{r_1} = Kq_1\sin(\vartheta_1 - \omega_1), \quad V_{\vartheta_1} = \frac{K}{p_1 r_1}, \qquad (2,8)$$

where r_1 is the value of the radius-vector at time of initial impulse.

Taking advantage of correlations (2,7), (2,8) and of dependences

$$r_1^{-1} = p^2 + pq\cos(\vartheta_1 - \omega) = p_1^2 + p_1 q_1\cos(\vartheta_1 - \omega_1), \qquad (2,9)$$

we transform the function (2,5) as follows:

$$g = \frac{q_1^2 + 3p_1^2}{2} + \frac{q^2}{2} - \frac{p^2}{2} - \frac{p_1^3}{p} - qq_1\cos(\omega_1 - \omega) + \\ + q_1\cos(\vartheta_1 - \omega_1)\left[2p_1 - p - \frac{p_1^2}{p}\right]. \qquad (2,10)$$

Effecting the differentiation, we obtain in the final form the system of necessary conditions:

$$q_1 \sin(\vartheta_1 - \omega_1)\left[2p_1 - p - \frac{p_1^2}{p}\right] + \lambda_1 [pq \sin(\vartheta_1 - \omega) - p_1 q_1 \sin(\vartheta_1 - \omega_1)] = 0, \tag{2,11}$$

$$\lambda_2 [pq \sin(\vartheta_2 - \omega) - p_2 q_2 \sin(\vartheta_2 - \omega_2)] = 0, \tag{2,12}$$

$$-p + \frac{p_1^3}{p^2} + q_1 \cos(\vartheta_1 - \omega_1)\left[\frac{p_1^2}{p^2} - 1\right] + \lambda_1 [2p + q\cos(\vartheta_1 - \omega)] + \\ + \lambda_2 [2p + q \cos(\vartheta_2 - \omega)] = 0, \tag{2,13}$$

$$q - q_1 \cos(\omega_1 - \omega) + \lambda_1 p \cos(\vartheta_1 - \omega) + \lambda_2 p \cos(\vartheta_2 - \omega) = 0, \tag{2,14}$$

$$q [- q_1 \sin(\omega_1 - \omega) + \lambda_1 p \sin(\vartheta_1 - \omega) + \lambda_2 p \sin(\vartheta_2 - \omega)] = 0, \tag{2,15}$$

$$p^2 + pq\cos(\vartheta_1 - \omega) - p_1^2 - p_1 q_1 \cos(\vartheta_1 - \omega_1) = 0, \tag{2,16}$$

$$p^2 + pq\cos(\vartheta_2 - \omega) - p_2^2 - p_2 q_2 \cos(\vartheta_2 - \omega_2) = 0. \tag{2,17}$$

The last system consists of seven equations with seven unknowns p, q, ω, ϑ_1, ϑ_2, λ_1, λ_2. The equations (2,12), (2,17) & (2,14), (2,15) may respectively substituted by the following ones, considering that in (2,14) and (2,15) $q \neq 0$:

$$pq \cos\omega = p_2 q_2 \cos\omega_2 + (p_2^2 - p^2)\cos\vartheta_2, \tag{2,18}$$

$$pq \sin\omega = p_2 q_2 \sin\omega_2 + (p_2^2 - p^2)\sin\vartheta_2, \tag{2,19}$$

$$q\cos\omega = q_1 \cos\omega_1 - p(\lambda_1 \cos\vartheta_1 + \lambda_2 \cos\vartheta_2), \tag{2,20}$$

$$q\sin\omega = q_1 \sin\omega_1 - p(\lambda_1 \sin\vartheta_1 + \lambda_2 \sin\vartheta_2). \tag{2,21}$$

<u>Remark 1</u>.- The multiplier λ_2 is $\neq 0$. In the opposite case the system of equations (2,11) — (2,17) would break up into two independent groups (2,11) — (2,16) and (2,17). The last equation (2,17) is the condition of flight orbit intersection with the final orbit. As to the first group of equations (2,11) — (2,16), it would define the transition to flight's intermediate orbit with minimum velocity accretion, without any kind of limitations. It may be shown, that such a minimum will be on the flight orbit coinciding with the initial orbit, but this case is excluded from the consideration as being trivial.

Remark 2.- Assume that p, q, ω are found. Then the equation (2,17) will give, generally speaking, two values for the angle ϑ_2. As to the equation (2,12), it is the condition for the coincidence of these two values, as is easy to verify. Therefore, tangency of the flight orbit with the final one will take place at the point ϑ_2, as this should be expected from the purport of the problem set up.

Remark 3.- It follows from the equations (2,11), (2,15) and (2,16) that the tangency of a noncircular flight orbit ($q \neq 0$) with the initial orbit at the point of initial impulse is only possible in the case when the lines of apsides of the three orbits coincide.

#3. - FLIGHT BETWEEN CIRCULAR AND SMALL ECCENTRICITY ORBITS WITHOUT TAKING INTO ACCOUNT THE TIME OF MOTION

In case of circular boundary orbits

$$q_1 = 0, \quad q_2 = 0. \tag{3,1}$$

From the equations (2,11) and (2,12), we have

$$\sin(\vartheta_1 - \omega) = 0, \quad \sin(\vartheta_2 - \omega) = 0, \tag{3,2}$$

since we are in a position to show that $\lambda_1 = 0$ does not satisfy the remaining equations. Rejecting the trivial case $\vartheta_1 = \vartheta_2$, we obtain from the inequalities (3,2) two possibilities:

$$\vartheta_1 = \omega, \quad \vartheta_2 = \omega + \pi, \tag{3,3}$$

$$\vartheta_1 = \omega - \pi, \quad \vartheta_2 = \omega. \tag{3,4}$$

Let us introduce dual signs : the upper ones for the case (3,3), the lower ones for the case (3,4). The correlations (2,16) and (2,17) will then give

$$p = \sqrt{\frac{p_1^2 + p_2^2}{2}}, \quad q = \pm \frac{p_1^2 - p_2^2}{\sqrt{2(p_1^2 + p_2^2)}}. \tag{3,5}$$

Since q > 0 always, we shall obtain in the case (3,3) the flight from the orbit of lesser radius to that of greater radius, i. e. $r_2 > r_1$, $p_1 > p_2$ and the opposite in the case (3,4), i. e. $r_2 < r_1$, $p_1 < p_2$.

The equation (2,15) is satisfied automatically. From the equations (2,13) and (2,14) we find

$$\lambda_1 = \frac{p_2^4 - p_1^3 p}{4p^4}, \quad \lambda_2 = \frac{p_1^3 (p_1 - p)}{4p^4}. \tag{3,6}$$

Passing to Keplerian elements in formulas (3,5), we have:

$$a = \frac{1}{2}(a_1 + a_2), \quad e = \pm \frac{a_2 - a_1}{a_2 + a_1}, \tag{3,7}$$

where indices 1 and 2 denote the respective reference to initial and final orbits.

Therefore, in the given case the Homan ellipse resulted to be the unique optimum orbit (with a precision to arbitrary choice of).

Let us consider now the flight between orbits of small eccentricity, introducing the small parameter, so that

$$q_1 = \varepsilon q_1', \quad q_2 = \varepsilon q_2'. \tag{3,8}$$

We shall seek the solution of the optimum problem, i. e. the solution of equations (2,11) — (2,17) in the form of series by powers ε and retain only the terms of the first order. We shall denote by strokes the coefficients of series at first powers of ε, and without strokes — the quantities taken on the Homan ellipse.

By the strength of the system (2,11) — (2,17), and introducing the denotations A, B, R, S, we obtain

$$\vartheta_1' - \omega' = \pm \frac{p_1 q_1}{\lambda_1 pq} \sin(\vartheta_1 - \omega_1)\left[\frac{p}{p_1} + \frac{p_1}{p} + \lambda_1 - 2\right] = A, \tag{3,9}$$

$$\vartheta_2' - \omega' = \mp \frac{p_2 q_2}{pq} \sin(\vartheta_2' - \omega_2) = B, \tag{3,10}$$

$$(2p \pm q)\lambda_1' + (2p \mp q)\lambda_2' = \cos(\vartheta_1 - \omega_1)\left[1 - \frac{p_1^2}{p^2}\right] q_1' \mp (\lambda_1 - \lambda_2) q' +$$

$$+ \left(1 + \frac{2p_1^3}{p^3} - 2\lambda_1 - 2\lambda_2\right) p' = R, \tag{3,11}$$

$$\pm p\lambda_1' \mp p\lambda_2' = \cos(\omega_1 - \omega) q_1' \mp (\lambda_1 - \lambda_2) p' - q' = S, \tag{3,12}$$

$$\pm \lambda_1 pq(\vartheta_1' - \omega') \mp \lambda_2 pq(\vartheta_2' - \omega') = q \sin(\omega_1 - \omega) q_1', \tag{3,13}$$

$$(2p \pm q)p' \pm pq' = p_1 q_1' \cos(\vartheta_1 - \omega_1), \tag{3,14}$$

$$(2p \mp q)p' \mp pq' = p_2 q_2' \cos(\vartheta_2 - \omega_2). \tag{3,15}$$

From the equalities (3,9), (3,10) and (3,13) we find

$$\operatorname{tg}\omega=\frac{p_1q_1'\left[2\left(p_1-p\right)+\lambda_1p\right]\sin\omega_1-pp_2q_2'\lambda_2\sin\omega_2}{p_1q_1'\left[2\left(p_1-p\right)+\lambda_1p\right]\cos\omega_1-pp_2q_2'\lambda_2\cos\omega_2}. \qquad (3{,}16)$$

The equations (3,14) — (3,15) will give

$$p'=\frac{p_1q_1'\cos\left(\vartheta_1-\omega_1\right)+p_2q_2'\cos\left(\vartheta_2-\omega_2\right)}{4p}, \qquad (3{,}17)$$

$$q'=-\frac{p_1q_1'\left(q\mp 2p\right)\cos\left(\vartheta_1-\omega_1\right)+p_2q_2'\left(q\pm 2p\right)\cos\left(\vartheta_2-\omega_2\right)}{4p^2}, \qquad (3{,}18)$$

while from the equations (3,11) and (3,12) we have:

$$\lambda_1'=\frac{pR-\left(q\mp 2p\right)S}{4p^2}, \quad \lambda_2'=\frac{pR-\left(q\pm 2p\right)S}{4p^2}. \qquad (3{,}19)$$

In the equation obtained by substraction of equations (2,11) and (2,12) from (2,15), we shall compute the terms of the second order relative to ;

$$\alpha\vartheta_1'+\beta\vartheta_2'+\gamma\omega'=C, \qquad (3{,}20)$$

where

$$\left.\begin{aligned} C&=q_1'\left[\frac{p_1^2-p^2}{p^2}p'-p_1\lambda_1'\mp q'\right]\sin\left(\vartheta_1-\omega_1\right)-p_2q_2'\lambda_2'\sin\left(\vartheta_2-\omega_2\right),\\ \alpha&=p_1q_1'\left\{\frac{p}{p_1}+\frac{p_1}{p}-2+\lambda_1\right\}\cos\left(\vartheta_1-\omega_1\right), \quad \beta=p_2q_2'\lambda_2\cos\left(\vartheta_2-\omega_2\right),\\ \gamma&=qq_1'\cos\left(\omega_1-\omega\right). \end{aligned}\right\} \qquad (3{,}21)$$

For ω', ϑ_1', ϑ_2' we shall find from (3,9), (3,10) and (3,20)

$$\omega'=\frac{-\alpha A-\beta B+C}{\alpha+\beta+\gamma}, \quad \vartheta_1'=\omega'+A, \quad \vartheta_2'=\omega'+B. \qquad (3{,}22)$$

Let us now compare the results obtained with those by Lawden [2] from a two-pulse flight between orbits. In this last paper the value of the characteristic velocity of the initial and final impulses (boosts) is minimized. The correction formulas for the eccentricity and the parameter are equivalent, i. e. the shape of the orbit is the same in both flights. In the expressions for tg ω the coefficients standing at trigonometric functions differ from the corresponding coefficients of formula (3,16).

#4.- MOTION TIME ALONG THE ORBIT

We shall find now the motion time along the orbit as a function of the polar angle ϑ. Assume that a polar angle ϑ_1 corresponds to the time t_1 and a polar angle ϑ_2 to the time t_2. We shall introduce the function

$$\psi = K(t_2 - t_1) = \psi(\vartheta_2, \vartheta_1, p, q, \omega), \qquad (4,1)$$

where K is given by the formula (2,6), and p, q, ω — by the elements of the orbit considered.

From the surface integral

$$r^2 \frac{d\vartheta}{dt} = Kp^{-1} \qquad (4,2)$$

it is easy to find that

$$\psi = p^{-1} \int_{\vartheta_1-\omega}^{\vartheta_2-\omega} \frac{dv}{(p+q\cos v)^2}. \qquad (4,3)$$

We shall compute the partial derivatives if the function

$$\frac{\partial\psi}{\partial\vartheta_2} = \frac{1}{p\,[p+q\cos(\vartheta_2-\omega)]^2}, \quad \frac{\partial\psi}{\partial\vartheta_1} = -\frac{1}{p\,[p+q\cos(\vartheta_1-\omega)]^2}, \qquad (4,4)$$

$$\frac{\partial\psi}{\partial\omega} = -\frac{\partial\psi}{\partial\vartheta_2} - \frac{\partial\psi}{\partial\vartheta_1}, \qquad (4,5)$$

$$\frac{\partial\psi}{\partial p} = -p^{-2} \int_{\vartheta_1-\omega}^{\vartheta_2-\omega} \frac{dv}{(p+q\cos v)^2} - 2p^{-1} \int_{\vartheta_1-\omega}^{\vartheta_2-\omega} \frac{dv}{(p+q\cos v)^3}, \qquad (4,6)$$

$$\frac{\partial\psi}{\partial q} = -2p^{-1} \int_{\vartheta_1-\omega}^{\vartheta_2-\omega} \frac{\cos v}{(p+q\cos v)^3}\,dv. \qquad (4,7)$$

Expanding the function ψ by by powers of eccentricity $e = \frac{q}{p}$ with a precision to the terms of the first order, we obtain

$$\psi = \frac{\vartheta_2 - \vartheta_1}{p^3} - \frac{2q}{p^4}\,[\sin(\vartheta_2-\omega) - \sin(\vartheta_1-\omega)]. \qquad (4,8)$$

We shall find the function ψ and its derivatives in case of flight along the Homan ellipse. Let ϑ_1 be the polar angle at time of initial boost, and ϑ_2 — the polar angle at time of encounter with the final orbit. Then we shall have

../..

$$\psi=\frac{\pi}{(p^2-q^2)^{3/2}}\,, \tag{4,9}$$

$$\left.\begin{aligned} &\frac{\partial\psi}{\partial\vartheta_2}=\frac{1}{p\,[p\mp q]^2}\,,\quad \frac{\partial\psi}{\partial\vartheta_1}=-\frac{1}{p\,[p\pm q]^2}\,,\quad \frac{\partial\psi}{\partial\omega}=\mp\frac{4q}{(p^2-q^2)^2}\,,\\ &\frac{\partial\psi}{\partial p}=-\frac{3p\pi}{(p^2-q^2)^{5/2}}\,,\quad \frac{\partial\psi}{\partial q}=\frac{3q\pi}{(p^2-q^2)^{5/2}}\,. \end{aligned}\right\} \tag{4,10}$$

#5. - SYSTEM OF NECESSARY CONDITIONS IN THE CASE OF FLIGHT TAKING INTO ACCOUNT THE CONCRETE MOTION TIME ALONG ORBITS

We shall consider two material bodies moving along the initial and final orbits. Depending upon the concrete physical problem, we can take for each material body either a rocket, a satellite or a planet. We shall neglect the masses of bodies relative to that of the central body, and also the dimensions of material bodies.

Assume that T_1 is the time of passage through the pericenter on the initial orbit with elements p_1, q_1, ω_1; obviously, at that time the polar angle of the first material body will be ω_1. To the moment T_2 of passage through the pericenter over the final orbit with elements p_2, q_2, ω_2 will correspond the polar angle ω_2.

At the time t_1, when the polar angle is equal to ϑ_1, there takes place a boost (impulse), as a result of which the first material body changes its orbit. At the time t_2, when the polar angle is ϑ_2, the intermediate orbit of the flight intersects the final orbit, and the material bodies meet.

We must find the flight orbit, i.e. the elements p, q, ω, and also the times t_1 and t_2 and the polar angles ϑ_1, ϑ_2 in such a way, that the value of the characteristic velocity of the initial boost at the time t_1 be minimum.

We shall use the equations (1,13) and (1,15) as the system of necessary conditions with the corresponding substitituion of denotations as was done in #2. The expression (2,10) is valid for the function g. The conditions (2,2) and (2,3) must be fulfilled at elements' discontinuity points. But besides that, it is necessary to add the conditions of motion time coincidence till the encounter of material bodies.

We shall introduce the following denotations :

$$\alpha = K(T_2 - T_1), \tag{5,1}$$
$$\psi_1 = K(t_1 - T_1) = \psi(\vartheta_1, \omega_1, p_1, q_1, \omega_1), \tag{5,2}$$
$$\psi = K(t_2 - t_1) = \psi(\vartheta_2, \vartheta_1, p, q, \omega), \tag{5,3}$$
$$\psi_2 = K(t_2 - T_2) = \psi(\vartheta_2, \omega_2, p_2, q_2, \omega_2). \tag{5,4}$$

From the identity
$$K(t_1 - T_1) + K(t_2 - t_1) - K(t_2 - T_2) - \alpha \equiv 0 \tag{5,5}$$
we shall obtain
$$\psi_3 = \psi_1 + \psi - \psi_2 - \alpha = 0. \tag{5,6}$$

The condition of motion time coincidence may be determined by a different method. Assume that t_0 is a certain moment of time; ω_0 is the polar angle of the first material body ; $\tilde{\omega}$ is the angular distance between the bodies at this time, computed from the first material body. Then it is necessary that

$$\psi(\vartheta_1, \omega_0, p_1, q_1, \omega_1) + \psi(\vartheta_2, \vartheta_1, p, q, \omega) - \psi\left(\vartheta_2, \omega_0 + \tilde{\omega}, p_2, q_2, \omega_2\right) = 0. \tag{5,7}$$

Remark 1.- It is assumed in this work that the encounter of material bodies takes place during the first convolution of flight orbit or that $\vartheta_2 - \vartheta_1 < 2\pi$.

Remark 2.- The angles ϑ_1, ϑ_2 must be measured continuously from the polar axis, and should not be bounded by frames $[0, 2\pi]$.

The system of necessary conditions in the final form will be

$$q_1 \sin(\vartheta_1 - \omega_1)\left[2p_1 - p - \frac{p_1^2}{p}\right] + \lambda_1\left[pq\sin(\vartheta_1 - \omega) - p_1 q_1 \sin(\vartheta_1 - \omega_1)\right] + \\ + \lambda_3\left(-\frac{\partial\psi_1}{\partial\vartheta_1} - \frac{\partial\psi}{\partial\vartheta_1}\right) = 0, \tag{5,8}$$

$$\lambda_2\left[pq\sin(\vartheta_2 - \omega) - p_2 q_2 \sin(\vartheta_2 - \omega_2)\right] + \lambda_3\left(\frac{\partial\psi_2}{\partial\vartheta_2} - \frac{\partial\psi}{\partial\vartheta_2}\right) = 0, \tag{5,9}$$

$$-p + \frac{p_1^3}{p^2} + q_1\cos(\vartheta_1 - \omega_1)\left[\frac{p_1^2}{p^2} - 1\right] + \lambda_1\left[2p + q\cos(\vartheta_1 - \omega)\right] + \\ + \lambda_2\left[2p + q\cos(\vartheta_2 - \omega)\right] + \lambda_3\frac{\partial\psi}{\partial p} = 0, \tag{5,10}$$

$$q - q_1\cos(\omega_1 - \omega) + \lambda_1 p\cos(\vartheta_1 - \omega) + \lambda_2 p\cos(\vartheta_2 - \omega) + \lambda_3\frac{\partial\psi}{\partial q} = 0, \tag{5,11}$$

$$-qq_1\sin(\omega_1 - \omega) + \lambda_1 pq\sin(\vartheta_1 - \omega) + \lambda_2 pq\sin(\vartheta_2 - \omega) + \lambda_3\frac{\partial\psi}{\partial\omega} = 0, \tag{5,12}$$

$$p^2 + pq\cos(\vartheta_1 - \omega) - p_1^2 - p_1 q_1\cos(\vartheta_1 - \omega_1) = 0, \tag{5,13}$$

$$p^2 + pq\cos(\vartheta_2 - \omega) - p_2^2 - p_2 q_2\cos(\vartheta_2 - \omega_2) = 0, \tag{5,14}$$

$$\psi_1 + \psi - \psi_2 - \alpha = 0. \tag{5,15}$$

The system thus obtained consists of eight equations with eight unknowns : $\vartheta_1, \vartheta_2, p, q, \omega, \lambda_1, \lambda_2, \lambda_3$.

#6.- FLIGHT BETWEEN CIRCULAR ORBITS TAKING INTO ACCOUNT THE MOTION TIME

In case of circular boundary orbits

$$q_1=0, \quad q_2=0. \tag{6,1}$$

Taking into account (4,8), we shall find

$$\psi_1=\frac{\vartheta_1-\omega_1}{p_1^3}, \quad \psi_2=\frac{\vartheta_2-\omega_2}{p_2^3}. \tag{6,2}$$

Effecting the substitution of the correlations (6,1) and (6,2) into the system of equations (5,8) — (5,15), we shall find, that it follows from the equation (5,11) that all $\lambda_1, \lambda_2, \lambda_3$ are not simultanously equal to zero. That is why the determinant, composed of coefficients at the multipliers $\lambda_1, \lambda_2, \lambda_3$ in the homogenous equations (5,8), (5,9), (5,12) must be equal to zero. Let us compute the indicated determinant:

$$p^2q^2\sin(\vartheta_1-\omega)\sin(\vartheta_2-\omega)\left(\frac{\partial\psi}{\partial\omega}+\frac{\partial\psi}{\partial\vartheta_2}+\frac{\partial\psi}{\partial\vartheta_1}+\frac{1}{p_1^3}-\frac{1}{p_2^3}\right)=0. \tag{6,3}$$

From the correlation (4,5) and the equality (6,3) it follows, that either $\sin(\vartheta_1-\omega)$, or $\sin(\vartheta_2-\omega)$ are equal to zero. We shall prove that in both cases

$$\lambda_3=0. \tag{6,4}$$

Let us assume the opposite, i.e. that $\lambda_3 \neq 0$. Assume that $\sin(\vartheta_1-\omega)=0$; then, it follows from the equation (5,8)

$$\frac{1}{p_1^3}=-\frac{\partial\psi}{\partial\vartheta_1}=\frac{1}{p\,[p+q\cos(\vartheta_1-\omega)]^2}=\frac{p}{p_1^4},$$

and hence $p = p_1$, which is in contradiction with the equation (5,13). Similarly, we may show that (6,4) follows from $\sin(\vartheta_2-\omega)=0$

The equality (6,4) implies that the system of equations (5,8) — (5,15) splits into two groups: (5,8) — (5,14) and (5,15). The first group of equations coincides with the necessary conditions of the optimum problem without taking into account the motion time (#2) and thus has a unique solution — the Homan ellipse. The equation (5,15) is the condition of motion time coincidence and can always be satisfied by the appropriate choice of angular distance of the pericenter ω.

Taking into account the correlations (4,9), (6,2), we have from (5,15)

$$\frac{\vartheta_1-\omega_1}{p_1^3}+\frac{\pi}{(p^2-q^2)^{3/2}}-\frac{\vartheta_2-\omega_2}{p_2^3}-\alpha=0. \tag{6,5}$$

Let us note that here ω_1 is the angle, responding to the position of the first material body on the initial orbit at the time T_1; ω_2 is the angle responding to the position of the second material body on the final orbit at the time T_2. If the boundary orbits have small eccentricities, it is natural to keep for ω_1, ω_2 the sense of angular distances of pericenters already in the approximation of the zero order relative to these eccentricities.

Passing to Keplerian elements, it is respectively found in the cases (3,3) and (3,4)

$$\left.\begin{aligned}\omega=\pi\,\frac{a_2\sqrt{a_2}-\left(\frac{a_1+a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}}+\frac{\omega_1a_1\sqrt{a_1}-\omega_2a_2\sqrt{a_2}+\alpha}{a_1\sqrt{a_1}-a_2\sqrt{a_2}}\\ \vartheta_1=\omega,\quad a_2>a_1,\end{aligned}\right| \tag{6,6}$$

$$\left.\begin{aligned}\omega=\pi\,\frac{a_1\sqrt{a_1}-\left(\frac{a_1+a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}}+\frac{\omega_1a_1\sqrt{a_1}-\omega_2a_2\sqrt{a_2}+\alpha}{a_1\sqrt{a_1}-a_2\sqrt{a_2}},\\ \vartheta_1=\omega-\pi,\quad a_2<a_1.\end{aligned}\right| \tag{6,7}$$

Taking the condition of motion time coincidence in the form (5,7), we would obtain in exactly the same way

$$\left.\begin{aligned}\omega=\omega_0+\pi\,\frac{a_2\sqrt{a_2}-\left(\frac{a_1+a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}}-\tilde{\omega}\frac{a_2\sqrt{a_2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}},\\ \vartheta_1=\omega,\quad a_2>a_1,\end{aligned}\right\} \tag{6,8}$$

$$\left.\begin{aligned}\omega=\omega_0+\pi\,\frac{a_1\sqrt{a_1}-\left(\frac{a_1+a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}}-\tilde{\omega}\frac{a_2\sqrt{a_2}}{a_1\sqrt{a_1}-a_2\sqrt{a_2}},\\ \vartheta_1=\omega-\pi,\quad a_2<a_1.\end{aligned}\right| \tag{6,9}$$

#7. - PERIOD OF OPTIMUM SITUATIONS' RECURRENCE

It follows from formulas (6,6) and (6,7) that a unique ω is to be found for every $\omega_1, \omega_2 \notin \alpha$. We shall take in formula (5,1)

$$T_1 = T_1^0 + n_1 \Pi_1 ; \quad T_2 = T_2^0 + n_2 \Pi_2 \qquad (7,1)$$

where Π_1, Π_2 are the revolution periods over the initial and final orbits, n_1, n_2 are whole numbers. T_1^0, T_2^0 are certain fixed values of moments of passage time through the pericenters. Therefore, for each n_1, n_2 one may compute ω corresponding to them. If the closest moments of time among themselves are chosen for T_1 and T_2, we shall obtain the least value of ω responding to these time moments.

Let us find the variation period of the optimum value (the period of polar angle ϑ_1 variation will be the same), which is tantamount to the determination of the time period in the course of which the otpimum situation will recur. We shall use the interplanetary flight terminology for the sake of brevity.

We shall analyze the case of circular motion, for example at $a_2 > a_1$. We shall continuously vary the initial data in formula (6,8) and see how ω will vary. By the strength of the correlations

$$\omega_0 = \omega_{00} + \frac{2\pi}{\Pi_1}(t - t_0), \quad \widetilde{\omega} = -\frac{2\pi}{\Pi_s}(t - t_0) + 2n\pi, \qquad (7,2)$$

where ω_{00} is the initial value of ω_0 ; Π_s is the synodical revolution period of the finite planet; n is the number of complete synodycal periods, we shall obtain from formula (6,8)

$$\omega = \omega_{00} + \pi \frac{a_2\sqrt{a_2} - \left(\frac{a_1 + a_2}{2}\right)^{\frac{3}{2}}}{a_1\sqrt{a_1} - a_2\sqrt{a_2}} + 2\pi \frac{a_2\sqrt{a_2}}{a_2\sqrt{a_2} - a_1\sqrt{a_1}} n. \qquad (7,3)$$

It follows from the last correlation that ω has one and only one optimum value in the course of the synodical revolution period of a finite planet.

Let t_0 be the opposition time at $a_2 > a_1$ or the time of lower conjunction at $a_1 > a_2$, and ω_0 — the planets' polar angle at that moment of time. From (6,8), (6,9) we shall have:

$$\omega = \omega_0 + \pi \frac{a_2\sqrt{a_2} - \left(\frac{a_1 + a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1} - a_2\sqrt{a_2}}, \quad \vartheta_1 = \omega, \; a_2 > a_1, \tag{7,4}$$

$$\omega = \omega_0 + \pi \frac{a_1\sqrt{a_1} - \left(\frac{a_1 + a_2}{2}\right)^{3/2}}{a_1\sqrt{a_1} - a_2\sqrt{a_2}}, \quad \vartheta_1 = \omega - \pi, \; a_2 < a_1. \tag{7,5}$$

Thus, the polar angles always differ by a constant angle from the opposition times at times of optimum position of the initial thrust (boost) [see formulas (7,4), (7,5)] (or from the moments of lower conjunctions if $a_1 > a_2$). The mean period of optimum situation recurrence is equal to the mean synodical period of revolution of the finite planet.

EXAMPLE.- Let us consider the flight from Earth to Mars. In the astronomical unit sustem $a_1 = 1$, $a_2 = 1.524$, whence $\omega = - 0.525\pi + \omega_0$, with the angle 0.525π corresponding to 96 mean solar days. Thus, at optimum flight to Mars, it is always necessary to take off 96 days before the times of Mars opposition. Such a moment took place on October 31st 1962. since Mars' opposition was on 4 February 1963. The mean period of optimum situation recurrence for the flight to Mars is equal to the mean synodical period of Mars rotation, i. e. 780 mean solar days.

#8.- FLIGHT BETWEEN ORBITS OF SMALL ECCENTRICITIES TAKING INTO ACCOUNT THE CONCRETE MOTION TIME

We shall assume that q_1 and q_2 are small. We shall introduce the small parameter ε according to formulas (3,11). We shall seek the solution of the optimum problem, i. e. the solution of equations (5,8) — — (5,15) in the form of series by powers ε. We shall designate by strokes the series' factors at first powers ε, and without strokes — the solution on Homan ellipse with the appropriately chosen ω (as was done previously (see (6,6)— (6,9)). We shall retain only the terms of the first order of smallness.

Taking into account the correlations (4,8) — (4,10), and introducing the denotations $\widetilde{A}, \widetilde{B}, \widetilde{C},$ we find

$$\vartheta_1' - \omega' = \pm \frac{1}{\lambda_1 pq}\left[p_1 q_1'\left(\frac{p}{p_1} + \frac{p_1}{p} + \lambda_1 - 2\right)\sin(\vartheta_1 - \omega_1) + \frac{p_1 - p}{p_1^4}\lambda_3'\right] = \widetilde{A}, \quad (8,1)$$

$$\vartheta_2' - \omega' = \mp \frac{1}{\lambda_2 pq}\left[\lambda_2 p_2 q_2' \sin(\vartheta_2 - \omega_2) + \frac{p - p_2}{p_2^4}\lambda_3'\right] = \widetilde{B}, \quad (8,2)$$

$$(2p \pm q)\lambda_1' + (2p \mp q)\lambda_2' = \left(1 + \frac{2p_1^3}{p^3} - 2\lambda_1 - 2\lambda_2\right)p' \mp (\lambda_1 - \lambda_2)q' +$$

$$+ \frac{3p\pi}{(p^2 - q^2)^{5/2}}\lambda_3' + \cos(\vartheta_1 - \omega_1)\left[1 - \frac{p_1^2}{p^2}\right]q_1', \quad (8,3)$$

$$\pm p\lambda_1' \mp p\lambda_2' = \mp(\lambda_1 - \lambda_2)p' - q' - \frac{3q\pi}{(p^2 - q^2)^{5/2}}\lambda_3' + \cos(\omega_1 - \omega)q_1', \quad (8,4)$$

$$\mp \lambda_1 pq(\vartheta_1' - \omega') \mp \lambda_2 pq(\vartheta_2' - \omega') = \pm \frac{4q}{(p^2 - q^2)^{5/2}}\lambda_3' + qq_1'\sin(\omega_1 - \omega), \quad (8,5)$$

$$(2p \pm q)p' \pm pq' = p_1 q_1' \cos(\vartheta_1 - \omega_1), \quad (8,6)$$

$$(2p \mp q)p' \mp pq' = p_2 q_2' \cos(\vartheta_2 - \omega_2), \quad (8,7)$$

$$\frac{p_1 - p}{p_1^4}\vartheta_1' + \frac{p - p_2}{p_2^4}\vartheta_2' \mp \frac{4q}{(p^2 - q^2)^{3}}\omega' = \frac{3\pi}{(p^2 - q^2)^{5/2}}(pp' - qq') +$$

$$+ \frac{2\sin(\vartheta_1 - \omega_1)q_1'}{p_1^4} - \frac{2\sin(\vartheta_2 - \omega_2)q_2'}{p_2^4} = \widetilde{C}. \quad (8,8)$$

The equations (8,6), (8,7) coincide with the equations (3,14), (3,15), and hence formulas (3,17), (3,18) follow directly for **p', q'**. The latter means, that the shape of the flight orbit will be the same as in the case of the problem without taking into account the motion time.

As a consequence of equations (8,1), (8,2) and (8,5), we have

$$\lambda_3' = \frac{p_1 q_1'\left(\frac{p}{p_1} + \frac{p_1}{p} + \lambda_1 - 2 \pm \frac{q}{p_1}\right)\sin(\vartheta_1 - \omega_1) + p_2 q_2' \lambda_2 \sin(\vartheta_2 - \omega_2)}{p_2^{-3} - p_1^{-3}}. \quad (8,9)$$

From the equations (8,1), (8,2) and (8,8) we shall have for

$$\vartheta_1' = \frac{\widetilde{A}(p_2^{-3} - pp_1^{-4}) - \widetilde{B}(p_2^{-3} - pp_2^{-4}) - \widetilde{C}}{p_2^{-3} - p_1^{-3}}, \quad (8,10)$$

$$\vartheta_2' = \frac{\widetilde{A}(p_1^{-3} - pp_1^{-4}) - \widetilde{B}(p_1^{-3} - pp_2^{-4}) - \widetilde{C}}{p_2^{-3} - p_1^{-3}}, \quad (8,11)$$

$$\omega' = \frac{\widetilde{A}(p_1^{-3} - pp_1^{-4}) - \widetilde{B}(p_2^{-3} - pp_2^{-4}) - \widetilde{C}}{p_2^{-3} - p_1^{-3}}. \quad (8,12)$$

Formulas (8,9) — (8,12) and (3,17), (3,18) provide the correction to the Homan ellipse flight orbit in the case when the initial and final orbits have small eccentricities.

Remark.- The initial angle of the pull at initial thrust is not clearly figured in the work. The tangent of this angle may be easily found as the ratio of the radial to transverse velocity accretion component.

In conclusion I convey my sincere gratitude to my leader Prof. V. S. Novoselov for his constant aid in the work.

**** THE END ****

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